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APPROXIMATE METHOD FOR SOLVING A TWO-DIMENSIONAL PROBLEM OF ELASTICITY THEORY

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A numerical-analytical method based on approximation by harmonic or biharmonic functions is proposed for solving a mixed two-dimensional problem of elasticity theory. This method allows one to decrease the geometric dimensionality of the boundary-value problem by reducing it to minimization of the boundary residual. The resultant approximate analytical solution satisfies all equations of elasticity theory.

General Scheme of the Method. Let Ω be a multidimensional multiply connected region in $\mathbb{R}^n$ limited by a hypersurface Γ. We consider the general boundary-value problem

$$LU(X) = 0, \quad X \in \Omega, \tag{1}$$

$$lU(X) = \Psi(Y), \quad Y \in \Gamma, \tag{2}$$

where L is the linear vector differential operator and $U(X)$ and $\Psi(Y)$ are the elements of certain functional spaces $R_1(\Omega)$ and $R_2(\Gamma)$ [1].

We formulate the method of expansion of the solution of the boundary-value problem (1), (2) in terms of nonorthogonal functions as follows. Let $\{\Psi_k(X)\}_{k=1}^{\infty}$ be a system of the vector functions Ψ_k, which satisfies the following conditions: each $\Psi_k(X)$ satisfies Eq. (1) in the domain Ω; a new function $l\Psi_k(Y)$, where l is the operator from the boundary condition (2), is defined on Γ for each basic function $\Psi_k(X)$; the system of functions $\{\Psi_k(X)\}_{k=1}^{\infty}$ is linearly independent, dense, and full in the spaces $C_4(\Gamma)$ or $L_2(\Gamma)$.

We find the coefficients a_k of the best [in terms of $C_4(\Gamma)$ or $L_2(\Gamma)$] expansion of the functions $\Psi(Y)$ with respect to the first N functions of the system $\{\Psi_k(X)\}_{k=1}^{\infty}$:

$$\Psi(Y) \approx \sum_{k=1}^{N} a_k^{(N)} l\Psi_k(Y).$$

Then the expression $U^N \approx \sum\limits_{k=1}^{N} a_k^{(N)}\Psi_k(X)$ can be considered as an approximate solution of problem (1), (2), which tends to the exact solution as $N \to \infty$ under the condition of problem correctness.

The novelty of the present work is that an algorithm of constructing global and local basic functions $\{\Psi_k(X)\}_{k=1}^{\infty}$, which satisfy the above conditions, is proposed. The method of obtaining the basic functions is applicable for all canonical equations of mathematical physics [2–4]. A reasonable choice of these basic functions, which can be orthonormalized, allows one to solve a wide range of problems of mechanics with arbitrary boundary conditions.

Solution of the Boundary-Value Problem of the Planar Theory of Elasticity. Determination of stresses and displacements in a homogeneous isotropic linearly elastic body in the state of planar strain

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reduces to the solution of the boundary-value problem for the following system of equations (Lamé equations [5]):

$$(\lambda + G)\frac{\partial \theta}{\partial x} + G\nabla^2 u + F_x = 0, \qquad (\lambda + G)\frac{\partial \theta}{\partial y} + G\nabla^2 v + F_y = 0, \tag{3}$$

where $(\lambda+G) = G/(1-2\mu)$, G and μ are the shear modulus and Poisson's ratio, respectively, $\theta = \partial/\partial x + \partial/\partial y$, and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In problems of the planar stressed state, λ is replaced by $\lambda_1 = 2\lambda G/(\lambda + 2G)$. In the two-dimensional domain Ω, we seek a regular solution of system (3), which satisfies the following relations at the boundary Γ:

$$l_1 u = \alpha_{11}\sigma_{n1} + \beta_{11}u + \alpha_{21}\sigma_{n2} + \beta_{21}v = \psi^1, \quad l_2 v = \alpha_{12}\sigma_{n1} + \beta_{12}u + \alpha_{22}\sigma_{n2} + \beta_{22}v = \psi^2. \tag{4}$$

Here α_{ij}, β_{ij}, and ψ^i $(i,\, j = 1,\, 2)$ are known functions of the point $Y \in \Gamma$ and σ_{ni} are the components of the stress vector:

$$\sigma_{n1} = \sigma_x \cos(n, x) + \tau_{xy}\cos(n, y), \quad \sigma_{n2} = \tau_{xy}\cos(n, x) + \sigma_y \cos(n, y).$$

Here $\sigma_x = \lambda\theta + 2\mu\partial u/\partial x$, $\sigma_y = \lambda\theta + 2\mu\partial v/\partial y$, and $\cos(n,x)$ and $\cos(n,y)$ are the direction cosines of the external normal to the boundary at the point $Y \in \Gamma$.

The specific cases of the boundary-value problem (3), (4) are the first and second classical boundary-value problems of elasticity theory, where either the stresses ($\beta_{ij} = 0$ and $\alpha_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol) or the displacements ($\beta_{ij} = \delta_{ij}$ and $\alpha_{ij} = 0$) are prescribed at the boundary Γ, the mixed problem in which the stresses are set at some portion of the contour Γ and the displacements are set at the remaining portion, and other boundary-value problems. Without loss of generality, we ignore in what follows the mass forces F_x and F_y [5].

It is known that the displacement vector $\boldsymbol{u} = \{u, v\}$, which satisfies Eqs. (3), can be constructed using one of the formulas of the general solutions, for example, using the formula of Papkovich–Neuber [5, 6], which has the following form:

$$u = \Phi_1 - 0.25(1-\mu)^{-1}\frac{\partial(\Phi_0 + x\Phi_1 + y\Phi_2)}{\partial x}, \qquad v = \Phi_2 - 0.25(1-\mu)^{-1}\frac{\partial(\Phi_0 + x\Phi_1 + y\Phi_2)}{\partial y}. \tag{5}$$

Here Φ_0, Φ_1, and Φ_2 are functions of the coordinates x and y, which satisfy the Laplace equation

$$\frac{\partial^2 \Phi_i}{\partial x^2} + \frac{\partial^2 \Phi_i}{\partial y^2} = 0, \qquad i = 0,\, 1,\, 2. \tag{6}$$

The problem of fullness and generality of the solution to (5) was considered by Ostrosablin and Senashov [6].

It can be easily seen that Eq. (6) allows a group of extension and transfer relative to independent variables and a group of extension relative to dependent variables. Therefore, according to [2, 7–9], invariant-group solutions of the Laplace equation can be sought, for example, in the form

$$\Phi_0 = (c_1 x + b)^\alpha \varphi(\eta), \qquad \eta = \frac{c_2 y + h}{c_1 x + b}, \tag{7}$$

where α, c_1, c_2, b, and h are arbitrary real numbers.

Substituting (7) into (6), we obtain

$$(c_1 x + b)^{\alpha-2}[(\eta^2 + D^2)\varphi'' - 2\eta(\alpha-1)\varphi' + \alpha(\alpha-1)\varphi] = 0, \tag{8}$$

where φ' and φ'' are the first and second derivatives with respect to η, $(c_1 x + b)^{\alpha-2} \neq 0$, and $D^2 = c_2^2/c_1^2$.

We seek a solution of the differential equation (8) in the form of the series

$$\varphi = \sum_{k=0}^{\infty} a_k \eta^k. \tag{9}$$

Substituting (9) into (8) and equating the series coefficients at equal powers of η, we find the recurrent formula

$$a_{k+2} = -\{(\alpha-k)(\alpha-k-1)/[D^2(k+2)(k+1)]\}a_k, \tag{10}$$

which allows us to express all even coefficients of series (9) in terms of a_0 and all odd coefficients in terms of a_1.

Thus, series (9) with coefficients determined by formula (10) and arbitrary values of a_0 and a_1 is the general solution of Eq. (8).

We consider the solutions of Eq. (8) that satisfy the conditions of uniqueness, continuity, and finiteness. It follows from formula (10) that for $a_k \neq 0$ we have $a_{k+2} = 0$ only if the constant α takes the values $\alpha = k$ and $\alpha = k+1$. If these conditions are satisfied, it is possible to obtain finite solutions of Eq. (8) in the form of the polynomials $P_k^\alpha(\eta)$.

For $k = 0$ and $k = 1$, we have $a_2 = [-\alpha(\alpha-1)/(2D^2)]a_0$ and $a_3 = [-(\alpha-1)(\alpha-2)/(6D^2)]a_1$. Let $\alpha = 1$; then we have $a_2 = 0$ and $a_3 = 0$. In this case, a_0 and a_1 can be arbitrary. We choose a_0 and a_1 as coefficients that form the initial basis. We obtain the following polynomials, which are the general solutions of Eq. (8) for different values of α: $P_{k=\overline{0,1}}^{\alpha=1} = a_0 + a_1\eta$, $P_{k=\overline{0,2}}^{\alpha=2} = a_0(1-\eta^2/D^2) + a_1\eta$ ($a_3 = a_4 = \ldots = 0$ for $\alpha = 2$), $P_{k=\overline{0,3}}^{\alpha=3} = a_0(1-3\eta^2/D^2) + a_1(\eta - \eta^3/(3D^2))$ ($a_4 = a_5 = \ldots = 0$ for $\alpha = 3$), and $P_{k=\overline{0,4}}^{\alpha=4} = a_0(1-6\eta^2/D^2 + \eta^4/D^4) + a_1(\eta - \eta^3/D^2)$ ($a_5 = a_6 = \ldots = 0$ for $\alpha = 4$), ... ($k = 0, 2, 3, \ldots, m$).

Using the principle of superposition of the solutions due to homogeneity and linearity of Eq. (6), we write the solution of the Laplace equation as

$$\Phi_0(x,y) = \sum_{\alpha=1}^{N} A_\alpha (c_{1\alpha}x + b_\alpha)^\alpha P_k^\alpha(\eta) = A_0 a_{00} + A_1(c_{11}x + b_1)(a_{01} + a_{11}\eta)$$
$$+A_2(c_{12}x + b_2)^2[a_{02}(1-\eta^2/D^2) + a_{12}\eta] + A_3(c_{13}x + b_3)^3[a_{03}(1-3\eta^2/D^2)$$
$$+a_{13}(\eta - \eta^3/(3D^2))] + \ldots + A_N(a_{1N}x + b_N)^N P_k^N(\eta), \tag{11}$$

where $\eta = (c_{2\alpha}y + h_\alpha)/(c_{1\alpha}x + b_\alpha)$ and A_α are arbitrary coefficients to be determined. The number of these coefficient depends on the method of solution of the boundary-value problem and on the estimate of accuracy of the approximate solution.

Note that the expressions for the polynomials $P_k^\alpha(\eta)$ depend on the choice of the parameters a_0 and a_1, as in the case of Legendre and Chebyshev polynomials. For each polynomial, these coefficients should be chosen so that the polynomials $P_k^\alpha(\eta)$ have the least difference from zero. In Eq. (11), they are designated by $a_{0\alpha}$ and $a_{1\alpha}$, since they can be different for each polynomial. The parameters c_1, c_2, b, and h in solution (11) can also be different for each polynomial $P_k^\alpha(\eta)$; therefore, they are denoted by $c_{1\alpha}$, $c_{2\alpha}$, b_α, and h_α. These parameters are chosen so that the system of linear equations, to which the initial problem is reduced, is not ill-posed. For all N and A_α, function (11) satisfies Eq. (6) in the domain Ω, but not the boundary conditions on Γ. Since only the polynomial solutions were determined, we ignored the solutions via transcendent functions (for example, for $\alpha = k = 0$ we have the solution $\varphi = C_1 + C_2 \arctan\eta$, etc.).

Other solutions of Eq. (6) can be obtained if we seek the invariant solution in the form $\Phi = (c_2y + h)^\alpha\varphi(\eta)$, where $\eta = (c_1x + b)/(c_2y + h)$.

Since the harmonic functions are found in the form of polynomials, we can easily find the conjugate harmonic functions using the Cauchy–Riemann conditions [5].

Another method of the formation of new solutions from some known solutions is described by Ovsyannikov [7]. This method is not related to finding finite transformations, but is applicable only in the case of linear homogeneous equations. For these equations, the solutions depending on parameters generate new solutions by means of differentiation with respect to these parameters. In addition, it is possible to obtain new solutions of differential equations if the group of continuous transformations generated by the operators X_i is known. Let $U_k = \varphi_k(x,y)$ ($k = \overline{1,m}$) be the solutions of a linear homogeneous system; then the functions

$$U_{kk} = X_i(U_k - \varphi_k(x,y))\Big|_{U_k = \varphi_k(x,y)}$$

also form the solution of the initial system of equations [7]. It is important that, for the whole system of polynomial functions found, this system is linearly independent and full in the space $C_4(\Gamma)$ or $L_2(\Gamma)$.

The proposed algorithm of formation of the basic functions allows one to expand the polynomial representations of the basic functions given in [5, 10, 11]. The resultant harmonic polynomials in Cartesian coordinates possess more convenient analytical and computational properties than, for example, the spherical functions.

Substituting the generalized polynomials (11) into the Papkovich–Neuber function (5), we find the unknown coefficients A_α from the boundary conditions (4) using any of the existing methods [1, 5, 12–14], for example, the method of weighted residuals or the variational method if the problem allows the variational formulation.

Reduction of the Two-Dimensional Problem of Elasticity Theory to the Boundary-Value Problem for a Biharmonic Equation. In the absence of mass forces in Ω, the introduction of the Airy stress function $\Psi(x,y)$ [5]

$$\sigma_x = \frac{\partial^2\Psi}{\partial x^2}, \qquad \sigma_y = \frac{\partial^2\Psi}{\partial y^2}, \qquad \tau_{xy} = -\frac{\partial^2\Psi}{\partial x\,\partial y} \tag{12}$$

allows one to reduce the mixed problem of elasticity theory to the boundary-value problem for a biharmonic equation

$$\nabla^4\Psi = \frac{\partial^4\Psi}{\partial x^4} + 2\,\frac{\partial^4\Psi}{\partial x^2\,\partial y^2} + \frac{\partial^4\Psi}{\partial y^4} = 0 \quad \text{in} \quad \Omega, \tag{13}$$

$$\sigma_{nx}\Big|_{\Gamma_1} = X(s), \quad \sigma_{ny}\Big|_{\Gamma_1} = Y(s), \quad u\Big|_{\Gamma_2} = u(s), \quad v\Big|_{\Gamma_2} = v(s), \tag{14}$$

where s is the variable length of the arc of the curve Γ with a piecewise-continuous external normal $n(s)$, $X(s)$, $Y(s)$, $u(s)$, and $v(s)$ are the projections of distributed forces specified on Γ_1 and displacements of the boundary points at the axis of the Cartesian coordinate system specified on Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$), and $\sigma_{nx}\Big|_\Gamma$ and $\sigma_{ny}\Big|_\Gamma$ are the projections of internal stresses at the boundary points of Ω onto these axes:

$$\sigma_{nx}\Big|_\Gamma = d\big(\partial\Psi/\partial y\big|_\Gamma\big)/ds = \partial^2\Psi/\partial s\,\partial y\big|_\Gamma, \qquad \sigma_{ny}\Big|_\Gamma = d\big(\partial\Psi/\partial x\big|_\Gamma\big)/ds = \partial^2\Psi/\partial s\,\partial x\big|_\Gamma.$$

In finding the regular solution (stress and displacement fields continuous in the closed domain Ω), we assume that the functions $X(s)$ and $Y(s)$ are continuous on Γ_1 and $u(s)$ and $v(s)$ are continuous and differentiable on Γ_2 everywhere except for the inflection points of the boundary Γ [5, 12].

Using the stress function found by solving the boundary-value problem (13), (14) and the above formulas, we can determine the stresses and displacements at all points of the closed domain Ω. The stresses in the domain Ω are expressed in terms of $\Psi(x,y)$ using formulas (12). The formulas in the explicit form (suitable for a computer manual), which express the field of displacements in a singly connected domain Ω in terms of the stress function, can be found in [12].

We find biharmonic polynomials using the algorithm described above. Let expression (7) be the solution of Eq. (13), where we use Ψ instead of Φ. Substituting (7) into (13), we obtain the ordinary differential equation

$$(c_1x+b)^{\alpha-4}[(\eta^4+2D^2\eta^2+D^4)\varphi'''' - 4(\alpha-3)(\eta^3+D^2\eta)\varphi''' + 2(\alpha-2)(\alpha-3)(3\eta^2+D^2)\varphi''$$
$$-4(\alpha-1)(\alpha-2)(\alpha-3)\eta\varphi' + \alpha(\alpha-1)(\alpha-2)(\alpha-3)\varphi] = 0, \tag{15}$$

where the primes denote derivatives with respect to η and $(c_1x+b)^{\alpha-4} \neq 0$.

Substituting (9) into (15) and equating the coefficients of the series at equal powers of η, we obtain the recurrent formula

$$a_{k+4} = a_{k+2}[-2k(k-1) + 4k(\alpha-3) - 2(\alpha-2)(\alpha-3)]/[D^2(k+3)(k+4)]$$
$$+a_k[-k(k-1)(k-2)(k-3) + 4k(k-1)(k-2)(\alpha-3) - 6k(k-1)(\alpha-2)(\alpha-3)$$
$$+4k(\alpha-1)(\alpha-2)(\alpha-3) - \alpha(\alpha-1)(\alpha-2)(\alpha-3)]/[D^4(k+1)(k+2)(k+3)(k+4)], \tag{16}$$

which allows one to express all even coefficients of series (9) in terms of a_0 and all odd coefficients in terms of a_1.

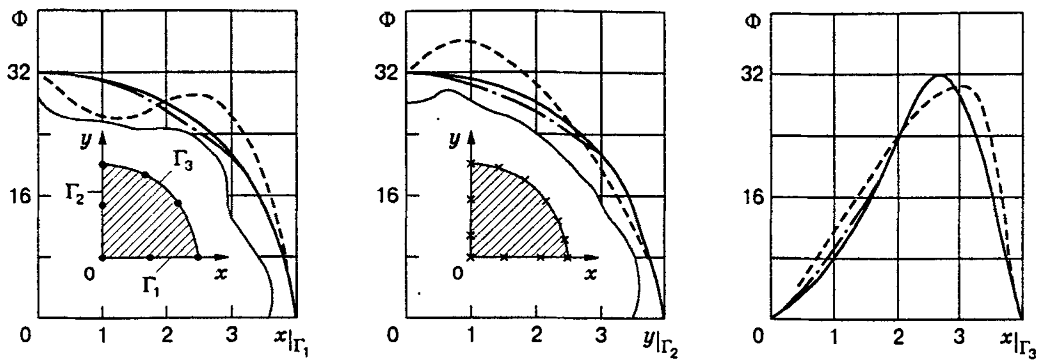

Fig. 1

The structure of Eq. (15) allows one to find solutions that satisfy the conditions of uniqueness, continuity, and finiteness. It follows from formula (16) that, for $a_k \neq 0$ and $a_{k+2} \neq 0$, we have $a_{k+4} = 0$ only if the constant α takes the values $\alpha = k+3$ and $\alpha = k+2$. If these conditions are fulfilled, it is possible to obtain finite solutions of Eq. (15) in the form of the polynomials $P_k^\alpha(\eta)$. It can be shown that for $k = 0$ and 1 and $\alpha = 3$, we have $a_4 = a_5 = a_6 = a_7 = \ldots = 0$. In this case, a_0, a_1, a_2, and a_3 can be arbitrary. We choose them as coefficients that form the initial basis. Using the principle of superposition of the solutions, we write the polynomial solution of the biharmonic equation (13) (similarly to the solution of the Laplace equation) in the expanded form

$$\Psi(x,y) = \sum_{\alpha=0}^{N} A_\alpha(c_{1\alpha}x + b_\alpha)^\alpha P_k^\alpha(\eta) = A_0a_{00} + A_1(c_{11}x + b_1)(a_{01} + a_{11}\eta)$$
$$+A_2(c_{12}x + b_2)^2[a_{02}(1-\eta^2) + a_{12}\eta] + A_3(c_{13}x + b_3)^3[a_{03} + a_{13}\eta + a_{23}\eta^2 + a_{33}\eta^3]$$
$$+A_4(c_{14}x + b_4)^4\Big[a_{04}\Big(1 - \frac{\eta^4}{D^4}\Big) + a_{14}\eta + a_{24}\Big(\eta^2 - \frac{\eta^4}{3D^2}\Big) + a_{34}\eta^3\Big] + A_5(c_{15}x + b_5)^5\Big[a_{05}\Big(1 - \frac{5\eta^4}{D^4}\Big)$$
$$+a_{15}\Big(\eta - \frac{\eta^5}{5D^4}\Big) + a_{25}\Big(\eta^2 - \frac{\eta^4}{D^2}\Big) + a_{35}\Big(\eta^3 - \frac{\eta^5}{5D^2}\Big)\Big] + A_6(c_{16}x + b_6)^6\Big[a_{06}\Big(1 - \frac{15\eta^4}{D^4} + \frac{2\eta^6}{D^6}\Big)$$
$$+a_{16}\Big(\eta - \frac{\eta^5}{D^4}\Big) + a_{26}\Big(\eta^2 - \frac{2\eta^4}{D^2} + \frac{\eta^6}{6D^4}\Big) + a_{36}\Big(\eta^3 - \frac{3\eta^5}{5D^2}\Big)\Big] + \ldots + A_N(c_{1N}x + b_N)^N P_k^N(\eta), \qquad (17)$$

where $\eta = (c_{2\alpha}y + h_\alpha)/(c_{1\alpha}x + b_\alpha)$.

It is important to note that function (17) satisfies Eq. (13) in the domain Ω for all N and A_α. Using the results presented above, we find other polynomial solutions of the biharmonic equation. The algorithm proposed allows one to considerably extend the polynomial representations of the basic functions for the biharmonic equation [5, 10, 12]. The boundary-value problem for a biharmonic equation relative to stress functions is solved in a similar manner.

Numerical Realization of the Method. Polynomial basic functions of the type (11) or (17) have the following advantages: they are not restricted to a specific domain Ω (the approximation of the solution is constructed only for the boundary Γ, since the polynomials are the exact solutions in the domain Ω) and ensure a uniform convergence of the approximate solutions to the exact solution both on the boundary Γ and in the domain Ω (according to the Weierstrass, Garnak, and Margelyan theorems [10, 12, 15–17]). These functions are convenient in computations. An important feature of the approach described is the possibility of evaluating the error of the approximate solution inside the domain, which is based on the difference in the boundary conditions.

As an example, we consider an approximate solution of Eq. (6), which satisfies the following boundary conditions at the quarter of the circumference $x^2 + y^2 = 16$ (boundary Γ): $\Phi(x,y) = 32 - x^4/8$ on Γ_1 ($y = 0$), $\Phi(x,y) = 32 - y^4/8$ on Γ_2 ($x = 0$), and $\Phi(x,y) = x^2y^2$ on Γ_3 ($y = (16 - x^2)^{1/2}$) [13]. Note that the exact solution of this problem is the function $\Phi(x,y) = x^2y^2 + [256 - (x^2+y^2)^2]/8$. We use the approximate solution (11), which satisfies the Laplace equation inside the domain. The results of the exact and approximate solutions at the boundary Γ ($\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$) for $a_{0\alpha} = a_{1\alpha} = c_{1\alpha} = c_{2\alpha} = 1$, $b_\alpha = h_\alpha = 6$, and different number of collocation points N are shown in Fig. 1. The analytical solution is shown by the solid curve, and the results obtained by the method of collocation by the dashed ($N = 7$) and dot-and-dashed curve ($N = 12$). The circles and crosses indicate the exact and approximate solutions, respectively. The accuracy of the approximate solution was estimated using the formula $\Delta = \max\left|\Phi(x_i,y_i) - \Phi_0(x_i,y_i)\right|_\Gamma \leqslant \varepsilon$, where $\varepsilon = 0.001$.

The method proposed not only makes it possible to obtain the solutions in an analytical form (we found polynomial solutions of the canonical equations of mathematical physics of elliptic, parabolic, and hyperbolic types, which depend on the spatial coordinates x, y, and z, and the time t [2–4]), but also significantly decreases the dimensionality of the algebraic system of equations relative to unknown coefficients as compared with the finite-difference and finite-element methods, since the approximation of the solution is constructed only at the boundary. Using the global and local basic functions found here, one can solve a wide range of both linear and nonlinear problems [18] (using, for example, linearization methods) of mechanics and mathematical physics. Biharmonic and harmonic polynomials can be applied to solve a number of problems of hydromechanics, electrostatics, and elasticity theory (in particular, torsion and bending of prismatic bodies, bending of membranes and plates).

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